Closing *Tuesday*: 2.5-6

Closing Friday: 2.7-8, 2.8

Today: 2.6 and 2.7

Note: $\sqrt{x^2} = x$, if $x \geq 0$, and
$\sqrt{x^2} = -x$, if $x < 0$.

Entry Task: Find

going to $-\infty$ going to zero

1. $\lim_{x\to\infty} 3^{(1-x)} + 2^{\left(1+\frac{1}{x}\right)} = 0 + 2^{(1+0)} = \boxed{2}$

2. $\lim_{x\to\infty} \frac{1+7e^{(3x)}}{2e^{x}+4e^{(3x)}} \cdot \frac{\frac{1}{e^{3x}}}{\frac{1}{e^{3x}}} = \lim_{x\to\infty} \frac{\frac{1}{e^{3x}}+7}{\frac{2}{e^{2x}}+4} = \frac{0+7}{0+4} = \boxed{7/4}$

3. $\lim_{x\to\infty} \frac{\left(2x+\sqrt{1+x^2}\right)}{(5+4x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{2+\frac{\sqrt{1+x^2}}{x}}{\frac{5}{x}+4} \overset{\text{SEE NOTE}}{=} \lim_{x\to\infty} \frac{2+\sqrt{\frac{1}{x^2}+1}}{\frac{5}{x}+4} = \frac{2+1}{4} = \boxed{3/4}$

4. $\lim_{x\to-\infty} \frac{\left(2x+\sqrt{1+x^2}\right)}{(5+4x)} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \frac{2+\frac{\sqrt{1+x^2}}{x}}{\frac{5}{x}+4} = \lim_{x\to-\infty} \frac{2-\sqrt{\frac{1}{x^2}+1}}{\frac{5}{x}+4} = \frac{2-1}{4} = \boxed{1/4}$

NOTE: For $x>0$, $\frac{\sqrt{1+x^2}}{x} = \frac{\sqrt{1+x^2}}{\sqrt{x^2}} = \sqrt{\frac{1+x^2}{x^2}} = \sqrt{\frac{1}{x^2}+1}$

For $x<0$, $\frac{\sqrt{1+x^2}}{x} = \frac{\sqrt{1+x^2}}{-\sqrt{x^2}} = -\sqrt{\frac{1+x^2}{x^2}} = -\sqrt{\frac{1}{x^2}+1}$

5. $\lim_{x\to\infty}\left(\sqrt{3+5x+4x^2}-2x\right)\dfrac{\sqrt{3+5x+4x^2}+2x}{\sqrt{3+5x+4x^2}+2x}$

$$= \lim_{x\to\infty} \frac{3+5x+4x^2-4x^2}{\sqrt{3+5x+4x^2}+2x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$$

ASIDE: SINCE $x>0$,

$$\frac{\sqrt{3+5x+4x^2}}{x} = \frac{\sqrt{3+5x+4x^2}}{\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{x}+5}{\sqrt{\frac{3+5x+4x^2}{x^2}}+2} = \lim_{x\to\infty}\frac{\frac{3}{x}+5}{\sqrt{\frac{3}{x^2}+\frac{5}{x}+4}+2}$$

$$= \frac{0+5}{\sqrt{0+0+4}+2} = \boxed{\frac{5}{4}}$$

Strategies to compute: $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right]$

Special note: If given two fractions, combine them (common denom).

Try plugging in the value:

1. **If denominator ≠ 0, done!**
2. **If denom = 0 & numerator ≠ 0,** the answer is -∞, +∞ or DNE. Examine the sign of the output from each side.
3. **If denom = 0 & numerator = 0,** Use algebra to simplify and cancel until either the numerator or denominator is not zero.

Strategy 1: Factor/Cancel
Strategy 2: Simplify Fractions
Strategy 3: Expand/Simplify
Strategy 4: Multiply by Conjugate (if you see radicals)

Strategies to compute: $\lim_{x \to \infty} f(x)$

Special note: Combine into one fraction (might need conjugate if given two terms involving a radical).

1. Is it a known limit?

$$\lim_{x \to \infty} \frac{1}{x^a} = 0, \text{if } a > 0; \quad \lim_{x \to \infty} e^{-x} = 0;$$

$$\lim_{x \to \infty} \ln(x) = \infty; \quad \lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2}.$$

2. Rewrite in terms of known limits:

Strategy 1: Multiply top/bottom by $\frac{1}{x^a}$, where *a* is the largest power.
Strategy 2: Multiply top/bottom by e^{-rx}.

Special note:
If *x* is positive, then $x = \sqrt{x^2}$.
If *x* is negative, then $x = -\sqrt{x^2}$.

Note:

After you complete the homework, you can attempt 30 more limit problems directly from old tests using the practice sheet that I compiled on my website.
Here is the direct link:
https://sites.math.washington.edu/~aloveles/Math124Fall2017/m124LimitsPractice.pdf
(you can also see full solutions there)
You can also look at any old midterm or old final for more practice.

Limits Practice

With the techniques we have developed, we can now evaluate many different types of limits. Below is a large collection of limit problems each pulled directly from the old exam archives. For each problem, evaluate the limit and give justification. See if you can do these quickly. Solutions are posted online.

1. $\lim_{x\to A} \frac{\frac{1}{x}-\frac{1}{A}}{x-A}$

2. $\lim_{x\to 8} \frac{\sqrt{x-4}+2}{x-3}$

3. $\lim_{x\to -2^-} \frac{|x+2|}{x^2+7x+10}$

4. $\lim_{x\to\infty} \left(x-\sqrt{x^2+4x}\right)$

5. $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

6. $\lim_{x\to 3} \frac{x^2-4x+3}{x^2+4x-21}$

7. $\lim_{x\to 2} \frac{\sqrt{x^2+4}-\sqrt{8}}{x^2-4}$

8. $\lim_{\theta\to 0} \frac{\sin(13\theta)}{4\theta}$

9. $\lim_{t\to\pi/2} \frac{\sin(t)+\sqrt{\sin^2(t)+2\cos^2(t)}}{2\cos^2(t)}$

10. $\lim_{x\to 0} \frac{x^3-8}{\cos^2(x)}$

11. $\lim_{t\to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right)$

12. $\lim_{r\to 0} \frac{3r^2}{3-\sqrt{9-r^2}}$

13. $\lim_{x\to 0} \frac{e^x \sin(3x) + 5\sin(3x)}{x}$

14. $\lim_{x\to 2^-} \frac{e^x}{2-x}$

15. $\lim_{x\to\infty} \left(x - \sqrt{x^2+3}\right)$

16. $\lim_{t\to 3} \frac{\frac{1}{3} - \frac{3}{t^2}}{t-3}$

17. $\lim_{x\to 2} \frac{\sqrt{x^2+12}}{(x-2)^2}$

18. $\lim_{x\to 0^+} \frac{1+x}{e^{x^2-x} - 1}$

22. $\lim_{x\to 3} \frac{\sqrt{x^2-9}}{\sqrt{2x-6}}$

23. $\lim_{x\to 0^-} \frac{|x-|x||}{|2x-|x||}$

24. $\lim_{x\to 0} \frac{\sin(2x)}{x}$

25. $\lim_{x\to 3^+} \frac{x+e^x}{(3-x)e^x}$

26. $\lim_{t\to 0} \left(\frac{1}{2t\sqrt{1+2t}} - \frac{1}{2t}\right)$

27. $\lim_{x\to\infty} \frac{3x^{10}+x^8+3}{x^{10}-3x^6+2}$

28. $\lim_{x\to 1} \frac{e^x+7}{(x-1)^3}$

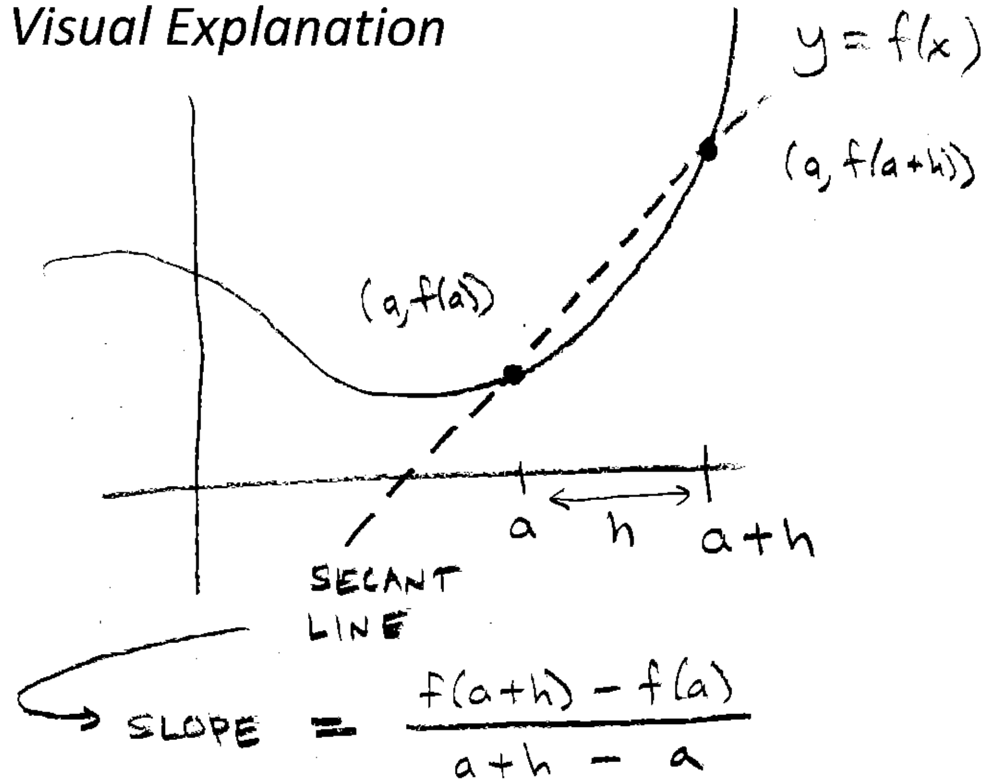
29. $\lim_{x\to 5} \frac{(x^2-25)\sin(x)}{(x-5)\cos(x)}$

30. $\lim_{x\to 0} \frac{x}{\sqrt{x+4}-2}$

2.7: The Derivative at a point

For a function, $y = f(x)$, we will define the ***slope of the tangent*** line to $f(x)$ at $x = a$ as follows:

Visual Explanation



Step 1: Let h represent an arbitrary small number. Draw the *secant* line through the graph at $x = a$ and $x = a + h$.

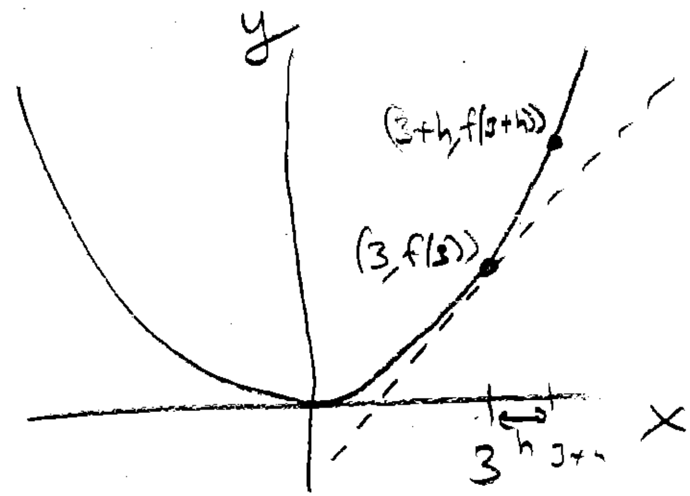
Step 2: Find the slope of this secant line (your answer will involve *h*).

Step 3: Find the limit as $h \to 0$, we will call this the slope of the tangent line at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Examples:

1. Find the slope of tangent line to $f(x) = x^2$ at $x = 3$. (Give the equation of the tangent line as well)



WE WANT

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{h(6+h)}{h}$$

$$= \boxed{6}$$

TANGENT LINE

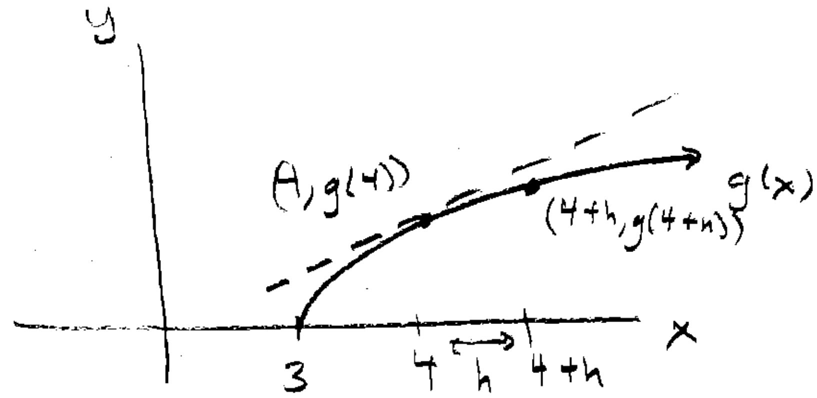
$f(3) = 9$

$f'(3) = 6$

$$\boxed{y = \underset{m}{6}(x - \underset{x_0}{3}) + \underset{y_0}{9}}$$

m ↑ $f'(3)$ y_0 ↑ $f(3)$

2. Find the equation for the tangent line to $g(x) = \sqrt{x-3}$ at $x = 4$.



WE WANT

$$g'(4) = \lim_{h \to 0} \frac{g(4+h) - g(4)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(4+h)-3} - \sqrt{4-3}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{(\sqrt{1+h}+1)}{(\sqrt{1+h}+1)}$$

$$= \lim_{h \to 0} \frac{1 + h - 1}{h(\sqrt{1+h}+1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{2}$$

TANGENT LINE:

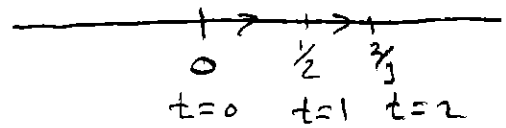
$g(4) = 1$, $g'(4) = \frac{1}{2}$

$$\boxed{y = \frac{1}{2}(x-4) + 1}$$

3. An object is moving on a number line. Assume the position of the object given by $s(t) = \frac{t}{t+1}$ feet.

(a) Find the instantaneous velocity at $t = 3$ seconds.

(b) Find the average velocity from $t = 3$ to $t = 4$ seconds.



$$\text{VELOCITY} = \frac{\Delta \text{LOCATION}}{\Delta \text{TIME}} = \frac{s(b) - s(a)}{b - a}$$

(END ↓ $s(b)$, START ↓ $s(a)$)

$$V(3) = s'(3) = \lim_{h \to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{3+h}{3+h+1} - \frac{3}{3+1}}{h} \cdot \frac{4(4+h)}{4(4+h)}$$

$$= \lim_{h \to 0} \frac{4(3+h) - 3(4+h)}{h(4)(4+h)}$$

$$= \lim_{h \to 0} \frac{12 + 4h - 12 - 3h}{h(4)(4+h)}$$

$$= \lim_{h \to 0} \frac{1}{4(4+h)} = \frac{1}{4(4+0)} = \boxed{\frac{1}{16} \frac{ft}{sec}}$$

NOTE: AVE VELOCITY FROM $t = 3$ TO $t = 3+h$ IS

$$\frac{s(3+h) - s(3)}{h} = \frac{1}{4(4+h)}$$

SO AVE VEL FROM $t = 3$ TO $t = 4$ WOULD BE WHEN $h = 1$

$$\frac{1}{4(4+1)} = \boxed{\frac{1}{20} \text{ ft/sec}}$$

Notes:

1. We call $f'(a)$ the **derivative** of $f(x)$ at $x = a$.

2. Graphically, $f'(a)$ is the **slope of the tangent line** to $y = f(x)$ at $x = a$.

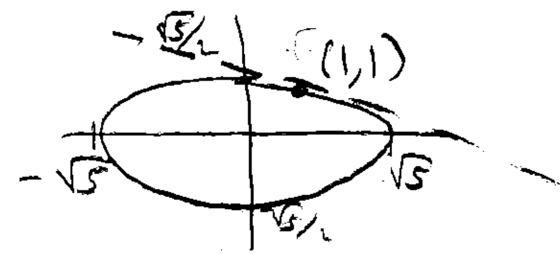
3. This is equivalent to saying $f'(a)$ is the **instantaneous rate of change** for $y = f(x)$ at $x = a$.

4. Given $y = f(x)$.
   Units of $f'(a)$ are $\frac{y\text{–}units}{x\text{–}units}$.

For example,
if $x =$ hours and $y = f(x) =$ miles,
then $f'(x) =$ miles/hour.

(Like HW!)

4. Find the equation for the tangent line to the ellipse $x^2 + 4y^2 = 5$ at the point $(x, y) = (1,1)$.



$$4y^2 = 5 - x^2 \Rightarrow y^2 = \frac{1}{4}(5 - x^2)$$

$$\Rightarrow y = \frac{1}{2}\sqrt{5 - x^2} = f(x)$$

WE WANT

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\sqrt{5-(1+h)^2} - \frac{1}{2}\sqrt{5-(1)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2}\sqrt{5-(1+h)^2} - 1}{h} \cdot \frac{\frac{1}{2}\sqrt{5-(1+h)^2} + 1}{\frac{1}{2}\sqrt{5-(1+h)^2} + 1}$$

$$= \lim_{h\to 0} \frac{\frac{1}{4}(5-(1+h)^2) - 1}{h\left(\frac{1}{2}\sqrt{5-(1+h)^2} + 1\right)}$$

$$= \lim_{h\to 0} \frac{\frac{1}{4}(5 - 1 - 2h - h^2) - 1}{h\left(\frac{1}{2}\sqrt{5-(1+h)^2} + 1\right)}$$

$$= \lim_{h\to 0} \frac{1 - \frac{1}{2}h - \frac{1}{4}h^2 - 1}{h\left(\frac{1}{2}\sqrt{5-(1+h)^2} + 1\right)}$$

$$\lim_{h\to 0} \frac{h\left(-\frac{1}{2} - \frac{1}{4}h\right)}{h\left(\frac{1}{2}\sqrt{5-(1+h)^2} + 1\right)}$$

$$= \frac{-\frac{1}{2} - 0}{\frac{1}{2}\sqrt{4} + 1}$$

$$= \frac{-\frac{1}{2}}{2} = \boxed{-\frac{1}{4}}$$

TANGENT LINE

$$\boxed{y = -\frac{1}{4}(x - 1) + 1}$$